Inflation and primordial perturbations

Problems

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Problems I

1. Show that for a massive scalar field with quadratic potential

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 \,,$$

   1.1 the potential slow-roll parameters are given by

$$\epsilon_v = \eta_v = \frac{1}{4\pi} \left(\frac{M_{Pl}}{\varphi} \right)^2 \,.$$

   1.2 Show that the slow-roll trajectory in phase-space is

$$\dot{\varphi}(\varphi) \simeq \frac{m M_P}{\sqrt{6\pi}} \mathrm{sign}(\varphi) \,.$$

2. Show that in the slow-roll approximation

$$H^{-1} \dot{\epsilon}_v \simeq 2\epsilon_v(-\eta_v + 2\epsilon_v) \,.$$

Problems II

3. Show that the primordial scalar power spectrum from single-field slow-roll inflation

$$P_R(k) \simeq \left[\left(\frac{H}{\dot{\varphi}}\right)^2 \left(\frac{H}{2\pi}\right)^2\right]_{k=aH}$$

can be written as

$$P_R(k) \simeq \left[\frac{H^2}{\pi \epsilon_v M_P^2}\right]_{k=aH}$$

Hence show that

$$n_R - 1 \equiv \frac{d\ln P_R}{d\ln k} \simeq -6\epsilon_v + 2\eta_v$$

Hint: use the answer to question 2, plus

$$H^{-1}\frac{d\ln H}{dt} = -\epsilon \quad , \quad H^{-1}\frac{d\ln(aH)}{dt} = 1-\epsilon$$

Problems III

4. Tensor metric perturbations can be decomposed into an integral over Fourier modes

$$h_{ij}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ h^{(+)}_{\vec{k}}(t) q^{(+)}_{\vec{k}\,ij} + h^{(\times)}_{\vec{k}}(t) q^{(\times)}_{\vec{k}\,ij} \right\} e^{i\vec{k}.\vec{x}} ,$$

where the *polarisation* tensors

$$\begin{aligned} q^{(+)}_{\vec{k}\,ij} &= \frac{1}{\sqrt{2}} \left(e_{\vec{k}\,i} e_{\vec{k}\,j} - \tilde{e}_{\vec{k}\,i} \tilde{e}_{\vec{k}\,j} \right) , \\ q^{(\times)}_{\vec{k}\,ij} &= \frac{1}{\sqrt{2}} \left(e_{\vec{k}\,i} \tilde{e}_{\vec{k}\,j} + \tilde{e}_{\vec{k}\,i} e_{\vec{k}\,j} \right) , \end{aligned}$$

$e_{\vec{k}\,i}$ and $\tilde{e}_{\vec{k}\,i}$ are orthonormal vectors ($e_{\vec{k}\,i} e^i_{\vec{k}} = \tilde{e}_{\vec{k}\,i} \tilde{e}^i_{\vec{k}} = 1$ and $e_{\vec{k}\,i} \tilde{e}^i_{\vec{k}} = 0$) and transverse to the wavevector $\vec{k}$.

Show that the polarisation tensors are

4.1 orthonormal

$$q^{(+)\,ij}_{\vec{k}} q^{(+)}_{\vec{k}\,ij} = q^{(\times)\,ij}_{\vec{k}} q^{(\times)}_{\vec{k}\,ij} = 1 , \quad q^{(+)\,ij}_{\vec{k}} q^{(\times)}_{\vec{k}\,ij} = 0 ,$$

4.2 tracefree

$$q^{(+)i}_{\vec{k}\;i} = q^{(\times)i}_{\vec{k}\;i} = 0 ,$$

4.3 and transverse to the wavevector $\vec{k}$:

$$k^i q^{(+)}_{\vec{k}\,ij} = k^i q^{(\times)}_{\vec{k}\,ij} = 0 .$$